

Title: Efficient Gram-matrix computation for irrational resonant systems using Kautz models

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Abstract

We present a new method for evaluating the Gram matrix containing the inner products of repeated integrals and derivatives of irrational transfer functions via a Kautz model. The technique is particularly interesting for describing resonant systems and it has an immediate application in model order reduction.

Keywords: Gram matrix; Model order reduction; Kautz functions; Resonant systems

1. Introduction

The Gram matrix, subsequently denoted as Ψ , contains the inner products of repeated integrals and/or derivatives of the impulse response of a system. It provides a description of the input-output behavior of the system and is therefore a potential tool in system identification and model order reduction. It is also closely related to controllability and observability Gramians [13]. A variety of methods permit a both fast and accurate computation of the Gram matrix of a rational system [2,7,12]. In the case of an irrational system an approximation of the successive derivatives and integrals is usually needed. In a previous paper [11] some of the authors proposed a method for Gram matrix computation through a rational approximation via Laguerre modeling. Laguerre functions have shown great potential in numerous applications (see [3] and its references). Nevertheless, when dealing with resonant, poorly damped systems, one may find this approach less satisfactory due to the poles of Laguerre functions which are restricted to the real axis. In this paper we propose a different approach based on Kautz functions. The latter can have complex conjugate poles and are therefore more adequate for modeling resonant systems. Consequently, their use in computing the Gram matrix of such systems can be a valuable alternative.

This communication is organized as follows. In section 2 we present a practical solution for computing the Kautz spectra of functions defined by their Laplace transforms and in section 3 the relations and properties binding the spectra of successive derivatives and integrals needed to compute the Gram matrix. To illustrate the applicability and efficiency of the technique two examples of model order reduction are given in section 4.

2. Kautz model

The two-parameter Kautz functions can be defined by their Laplace transforms [5,6,15]

$$\begin{aligned}\hat{\phi}_{2n}(s) &= \frac{\sqrt{2bc}}{s^2 + bs + c} \left(\frac{s^2 - bs + c}{s^2 + bs + c} \right)^n \\ \hat{\phi}_{2n+1}(s) &= \frac{\sqrt{2b} s}{s^2 + bs + c} \left(\frac{s^2 - bs + c}{s^2 + bs + c} \right)^n\end{aligned}\quad (1)$$

with $n = 0, 1, 2, \dots$ and $b, c \in \Re^{+*}$. The time domain functions are written as $\phi_m(t)$ or $\phi_m(t, b, c)$ whenever it is desirable to observe the parameters. The orthonormal set $\{\phi_m\}_{m \geq 0}$ is complete in $L^2[0, \infty[$, thus any finite-energy real-causal signal $g(t)$ can be approximated within any desired accuracy by truncating its infinite expansion

$$g(t) = \sum_{m=0}^{\infty} \gamma_m \phi_m(t) \quad (2)$$

where γ_m is the $m+1^{\text{th}}$ Kautz-Fourier coefficient, given by the inner product $\gamma_m = \langle g, \phi_m \rangle = \int_0^{\infty} g(t) \phi_m(t) dt$.

Our main objective in this section is to determine a set of relations allowing us to efficiently compute the Kautz spectrum of a function with the sole a priori knowledge of its Laplace transform. We use the pioneering results described in [15] and combine two known transformations in order to achieve it.

In 1996 Wahlberg and Makila [15], have shown that the set of coefficients $\{\gamma_m\}_{m \geq 0}$, can be found from power series calculations in the following manner. Denoting by $\hat{g}(s)$ the Laplace transform of $g(t)$, assumed to be analytic outside an appropriate region in the s -plane, let us decompose $\hat{g}(s)$ into two orthogonal partitions

$$\hat{g}(s) = \frac{\sqrt{c}}{s} \hat{g}^e(s) + \hat{g}^o(s). \quad (3)$$

Let $p = e, o$, each of the two partitions can then be expressed as

$$\hat{g}^p(s) = \sum_{n=0}^{\infty} \gamma_n^p \hat{\phi}_{2n+1}(s) \text{ with } \gamma_n^e = \gamma_{2n}, \gamma_n^o = \gamma_{2n+1}. \quad (4)$$

Using the fact that $\hat{g}^p(c/s) = \hat{g}^p(s)$ derived from (4), it follows from (3) that

$$\hat{g}^e(s) = \frac{\sqrt{c} s [\hat{g}(c/s) - \hat{g}(s)]}{s^2 - c}, \quad \hat{g}^o(s) = \frac{[s^2 \hat{g}(s) - c \hat{g}(c/s)]}{s^2 - c}. \quad (5)$$

Moreover, since $\hat{g}^p(c/s) = \hat{g}^p(s)$ ($p = e, o$) both of these functions are symmetric of s and c/s so they can be represented as functions of $s + (c/s)$ and $s(c/s) = c$. Denoting the transformation $s + (c/s) = \xi$, it follows that

$$\hat{g}^p(s) = G^p(s + c/s, c) \text{ with}$$

$$G^p(\xi, c) = \sum_{n=0}^{\infty} \gamma_n^p \hat{l}_n(\xi) \quad (p = e, o) \quad (6)$$

where $\hat{l}_n(\xi) = \frac{\sqrt{2b}}{\xi+b} \left(\frac{\xi-b}{\xi+b} \right)^n$ denotes the Laplace transform of the normalized Laguerre function $l_n(t, b)$.

Notice that the transformation $s \rightarrow s + (c/s)$ is common in filter design where it is used to design a band-pass filter from a low-pass filter.

A well-known relation binds the Laplace transform and the z-transform of the Laguerre spectrum: carrying out the bilinear transformation $\xi = b \frac{z+1}{z-1}$ in (6), one can deduce

$$\Gamma^p(z) = \sqrt{2b} \frac{z}{z-1} G^p \left(b \frac{z+1}{z-1} \right) \quad (p = e, o) \quad (7)$$

where

$$\Gamma^p(z) = \sum_{n=0}^{\infty} \gamma_n^p z^{-n} \quad (p = e, o). \quad (8)$$

The combination of the two transformations $s + (c/s) = \xi$ and $\xi = b \frac{z+1}{z-1}$ corresponds to the second-order all-pass transformation

$$z = \frac{s^2 + bs + c}{s^2 - bs + c}.$$

The inverse transformation has two solutions $s = a_{\pm}(z)$ where

$$a_{\pm}(z) = \frac{1}{2} \left(b \frac{z+1}{z-1} \pm \sqrt{\left(b \frac{z+1}{z-1} \right)^2 - 4c} \right) \quad (9)$$

with the properties $a_+(z)a_-(z) = c$ and $a_+(z) + a_-(z) = b \frac{z+1}{z-1}$.

From (9), (7) and (5) one can then deduce the following relation binding the Laplace transform and the z-transform of the Kautz spectrum

$$\begin{aligned} \Gamma^e(z) &= \sqrt{2b} \frac{z}{z-1} \left[\frac{\hat{g}(a_-(z)) - \hat{g}(a_+(z))}{a_+(z) - a_-(z)} \right], \\ \Gamma^o(z) &= \sqrt{2b} \frac{z}{z-1} \left[\frac{a_+(z)\hat{g}(a_+(z)) - a_-(z)\hat{g}(a_-(z))}{a_+(z) - a_-(z)} \right]. \end{aligned} \quad (10)$$

It will be noted that relations (10) are independent of the choice of the solution $s = a_+(z)$ or $s = a_-(z)$. One can find them very interesting in deriving the properties of Kautz spectra, despite the irrational form of $a_{\pm}(z)$. Moreover, relations (10) may also be used to numerically evaluate the Kautz spectrum of a given function from its Laplace transform: setting $z = e^{-j\theta}$ in (10) with $\theta \in [0, 2\pi[$, keeping in mind (8), the Kautz coefficients γ_m could then be computed using a FFT or DCT algorithm. Our first objective has thus been achieved.

3. Derivatives and integrals computation

Let $\Omega = \{g_1, g_2, \dots, g_q, \dots, g_{r+1}\}$ denote an orderly set of real functions where

$$\begin{aligned} g_q(t) &\triangleq g(t) \\ g_{i+1}(t) &\triangleq \frac{dg_i(t)}{dt} \quad (i = q, q+1, \dots, r) \\ g_{i-1}(t) &\triangleq \int_{-\infty}^t g_i(\tau) d\tau \quad (i = q, q-1, \dots, 2) \end{aligned} \quad (11)$$

$g(t)$ being the impulse response of the stable system with transfer function $\hat{g}(s)$ (assuming that $g(t)$ is a well-behaved square-integrable other responses of the system could be considered for example, the step response shifted of its final value).

The elements $\psi_{i,j}$ of the Gram matrix Ψ are all the possible inner products of real functions $g_i(t)$ and $g_j(t)$ defined as

$$\psi_{i,j} = \langle g_i, g_j \rangle \triangleq \int_0^\infty g_i(t) g_j(t) dt. \quad (12)$$

Derivative and integral operators defined in (11) preserve natural frequencies or poles in the Laplace domain; therefore Ω constitutes an efficient set of approximation functions for determining an r-order reduced model. When dealing with irrational transfer functions, direct computation of their Gram matrix could be an insurmountable problem. In this section we propose a solution based on Kautz modeling.

Let us consider the derivative operator defined in (11), in Laplace domain it yields

$$\hat{g}_{i+1}(s) = s\hat{g}_i(s) - g_i(+0). \quad (13)$$

With (9), using the fact that $[a_\pm(z)]^2 = b \frac{z+1}{z-1} a_\pm(z) - c$ one can deduce from (10)

$$\begin{aligned} \Gamma_{i+1}^e(z) &= -\sqrt{c}\Gamma_i^o(z) \\ \Gamma_{i+1}^o(z) &= b \frac{z+1}{z-1} \Gamma_i^o(z) + \sqrt{c}\Gamma_i^e(z) - \sqrt{2b} \frac{z}{z-1} g_i(+0). \end{aligned}$$

It follows that the Kautz spectra of the successive derivatives of $g_q(t)$ can be recursively computed using the following relations for $i = q, q+1, \dots, r$

$$\begin{aligned} \gamma_{i+1,n}^e &= -\sqrt{c}\gamma_{i,n}^o, \quad n = 0, 1, 2, \dots \\ \gamma_{i+1,n}^o &= \gamma_{i+1,n-1}^o + \sqrt{c}(\gamma_{i,n}^e - \gamma_{i,n-1}^e) + b(\gamma_{i,n}^o + \gamma_{i,n-1}^o), \quad n = 1, 2, \dots \end{aligned} \quad (14)$$

with $\gamma_{i+1,0}^o = \sqrt{c}\gamma_{i,0}^e + b\gamma_{i,0}^o - \sqrt{2b}g_i(+0)$ for $n = 0$. When the value $g_i(+0)$ is not known it can be computed as

$$g_i(+0) = \lim_{s \rightarrow \infty} s\hat{g}_i(s) = \sqrt{2b} \sum_{n=0}^{\infty} \gamma_{i,n}^o. \quad (15)$$

In practice, Kautz series are often truncated at order $M=2N$. Therefore, and more particularly when the initial values $g_i(+0)$ are not readily available, an approximation of the coefficients of the various Kautz spectra can be

computed for $i = q, q+1, \dots, r$ as follows

$$\begin{aligned}\tilde{\gamma}_{i+1,n}^e &= -\sqrt{c}\tilde{\gamma}_{i,n}^o, \quad n = N-1, N-2, \dots, 0 \\ \tilde{\gamma}_{i+1,n}^o &= \tilde{\gamma}_{i+1,n+1}^o - \sqrt{c}(\tilde{\gamma}_{i,n+1}^e - \tilde{\gamma}_{i,n}^e) - b(\tilde{\gamma}_{i,n+1}^o + \tilde{\gamma}_{i,n}^o), \quad n = N-2, N-3, \dots, 0\end{aligned}\quad (16)$$

with $\tilde{\gamma}_{i+1,N-1}^o = \sqrt{c}\tilde{\gamma}_{i,N-1}^e - b\tilde{\gamma}_{i,N-1}^o$ for $n = N-1$ and where the starting point of the computation is $\tilde{\gamma}_{q,n}^p = \gamma_{q,n}^p$ ($p = e, o$). The approximate Kautz series of $g_i(t)$ is then given by

$$\tilde{g}_i(t) = \sum_{n=0}^{N-1} \tilde{\gamma}_{i,n}^e \varphi_{2n}(t) + \sum_{n=0}^{N-1} \tilde{\gamma}_{i,n}^o \varphi_{2n+1}(t). \quad (17)$$

An approximated value of $g_i(+0)$ can then be evaluated using (15) or (14) i.e.

$$\tilde{g}_i(+0) = \sqrt{2b} \sum_{n=0}^{N-1} \tilde{\gamma}_{i,n}^o = \frac{1}{\sqrt{2b}} (\sqrt{c}\tilde{\gamma}_{i,0}^e + b\tilde{\gamma}_{i,0}^o - \tilde{\gamma}_{i+1,0}^o). \quad (18)$$

Integral and derivative operators as defined in (11) are reciprocal; we can therefore obtain the recurrence relations binding the Kautz spectra of the successive integrals of $g(t)$ directly from (14) for $i = q, q-1, \dots, 2$

$$\begin{aligned}\gamma_{i-1,n}^o &= -\frac{1}{\sqrt{c}}\gamma_{i,n}^e, \quad n = 0, 1, 2, \dots \\ \gamma_{i-1,n}^e &= \gamma_{i-1,n-1}^e + \frac{1}{\sqrt{c}}(\gamma_{i,n}^o - \gamma_{i,n-1}^o) + \frac{b}{c}(\gamma_{i,n}^e + \gamma_{i,n-1}^e), \quad n = 1, 2, \dots\end{aligned}\quad (19)$$

with $\gamma_{i-1,0}^e = \frac{b}{c}\gamma_{i,0}^e + \frac{1}{\sqrt{c}}\gamma_{i,0}^o + \sqrt{\frac{b}{2c}}g_{i-1}(+0)$ for $n = 0$.

When working with truncated Kautz series at order $M=2N$ or when facing difficulties to calculate $g_{i-1}(+0)$ it is easier to use the following expressions:

$$\begin{aligned}\tilde{\gamma}_{i-1,n}^o &= -\frac{1}{\sqrt{c}}\tilde{\gamma}_{i,n}^e, \quad n = N-1, N-2, \dots, 0 \\ \tilde{\gamma}_{i-1,n}^e &= \tilde{\gamma}_{i-1,n+1}^e - \frac{1}{\sqrt{c}}(\tilde{\gamma}_{i,n+1}^o - \tilde{\gamma}_{i,n}^o) - \frac{b}{c}(\tilde{\gamma}_{i,n+1}^e + \tilde{\gamma}_{i,n}^e), \quad n = N-2, N-3, \dots, 0\end{aligned}\quad (20)$$

with $\tilde{\gamma}_{i-1,N-1}^e = \frac{1}{\sqrt{c}}\tilde{\gamma}_{i,N-1}^o - \frac{b}{c}\tilde{\gamma}_{i,N-1}^e$ for $n = N-1$ and where the starting point of the computation is $\tilde{\gamma}_{q,n}^p = \gamma_{q,n}^p$ ($p = e, o$).

Keeping in mind that Kautz functions form an orthonormal basis, the inner products defined by (12) can therefore be expressed as

$$\psi_{i,j} = \sum_{n=0}^{\infty} \gamma_{i,n}^e \gamma_{j,n}^e + \sum_{n=0}^{\infty} \gamma_{i,n}^o \gamma_{j,n}^o \quad (21)$$

and all elements of the Gram matrix can at this point be computed. Using the properties of the Gram matrix given in [11], an efficient algorithm for its computation is summarized in the appendix.

The recurrent relations described in this section allow an efficient way to compute the Kautz model of the successive derivatives and integrals of a function. Moreover, equations (16) and (20) yield two sets of functions

verifying the strict equalities $\tilde{g}_{i+1}(t) = d\tilde{g}_i(t)/dt$ and $\tilde{g}_{i-1}(t) = \int_{-\infty}^t \tilde{g}_i(\tau) d\tau$ respectively. Consequently, in model order reduction applications, at least two reduced order models provably stable [9] can be readily obtained according to the procedure described in [14] for any given value of q .

Especially in the case of resonant systems, with their two parameters granting them an extra degree of freedom, Kautz functions should allow either a more accurate computation of the Gram matrix, for an equal number of coefficients N or a faster way of computation of the Gram matrix than their Laguerre counterparts, for a given precision.

A good choice of Kautz parameters may contribute to improve the efficiency or accuracy of the computation (the same remark could be made for the choice of the Laguerre parameter). Different works addressed the problem and proposed optimal or suboptimal solutions for making this choice [1,8,10].

4. Examples

To illustrate the proposed method, the Gram matrices of two infinite-dimensional resonant systems will be computed and used in a typical model order reduction application.

Let us first consider a simple closed-loop system with a delay of $T = 1$ sec in the feedback (Fig.1.).

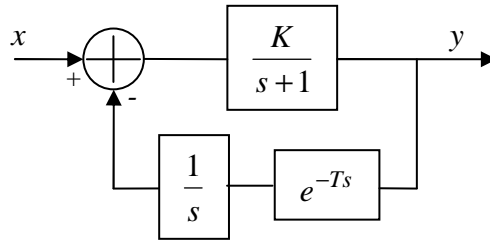


Fig.1. A closed-loop system with a delay in the feedback

This system has an infinite order transfer function given by

$$\hat{g}(s) = \frac{Ks}{s^2 + s + Ke^{-Ts}} \quad (T = 1 \text{ sec})$$

which becomes unstable when K grows beyond a certain value ($K \approx 1.135$ for $T = 1$ sec). Taking $K = 1$, the impulse response given in Fig.2. shows strong but decreasing oscillations. The first 10 Kautz coefficients are computed according to the technique described in section 2. We chose $c = 0.68$ and then computed a quasi-optimal corresponding value of $b = 0.088$ minimizing a superior boundary of the truncation error [8]. As we search for order $r = 3$ rational models, a size $(r+1) \times (r+1)$ Gram matrix is computed following the algorithm given in the appendix. A set of reduced order 3 models, guaranteed to be stable, can readily be derived using the method described in [14] for $q = 1, 2, \dots, r+1$. The best model $\tilde{g}_K(s)$ with respect to the relative quadratic error $\|\mathcal{E}\|^2 = \int_0^\infty [g(t) - \tilde{g}(t)]^2 dt / \int_0^\infty [g(t)]^2 dt$ is obtained for a Gram matrix constructed with the inner products of $g(t)$, its first integral and its first two derivatives (i.e. $q = 2$):

$$\tilde{g}_K(s) = \frac{0.9388s^2 + 2.159s + 0.00436}{s^3 + 3.122s^2 + 0.9325s + 2.051}.$$

The corresponding relative quadratic error is $\|\varepsilon_K\|^2 = 4.44 \times 10^{-3}$. For this system, a good approximation of the transfer function can also be obtained by replacing the delay in the feedback by its Padé approximant $e^{-Ts} \approx (2 - Ts)/(2 + Ts)$ which leads to

$$\tilde{g}_P(s) = \frac{s^2 + 2s}{s^3 + 3s^2 + s + 2}.$$

It yields a relative quadratic error of $\|\varepsilon_P\|^2 = 7.12 \times 10^{-2}$. The previous model $\tilde{g}_K(s)$ is clearly more precise benefiting from a very accurate computation of the Gram matrix with a relatively small number of Kautz coefficients. Due to the highly resonant nature of the system, achieving such accuracy via a Laguerre decomposition as described in [11] would require a much greater number of coefficients (several hundred for a well chosen Laguerre parameter). For the sake of comparison we specify that the use of 10 Laguerre coefficients would in this case lead to a relatively bad model with a respective quadratic error $\|\varepsilon_L\|^2 = 1.44 \times 10^{-1}$. This underlines the advantages of computing the Gram matrix via Kautz functions in applications dealing with resonant systems. The original impulse response and those of the three proposed models can be compared in Fig. 2 and 3.

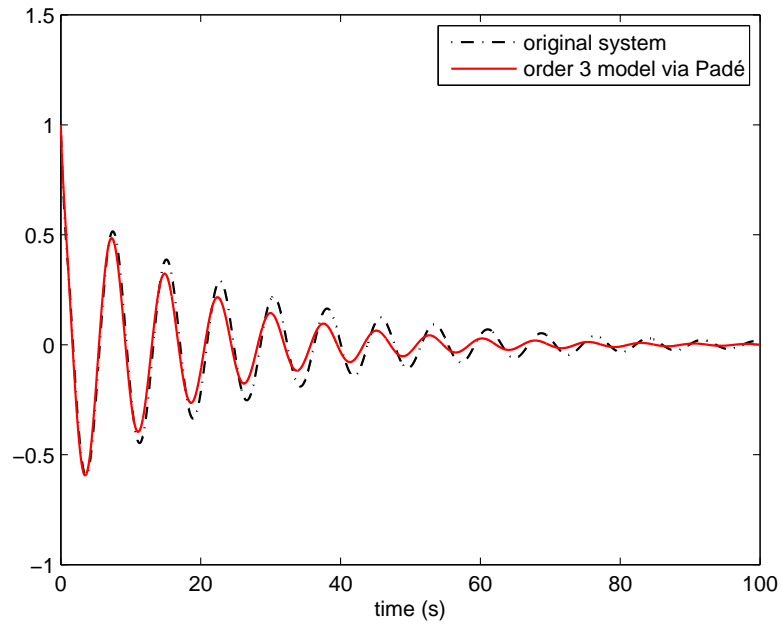


Fig.2. Impulse response of the original system and the order 3 model obtained via Padé

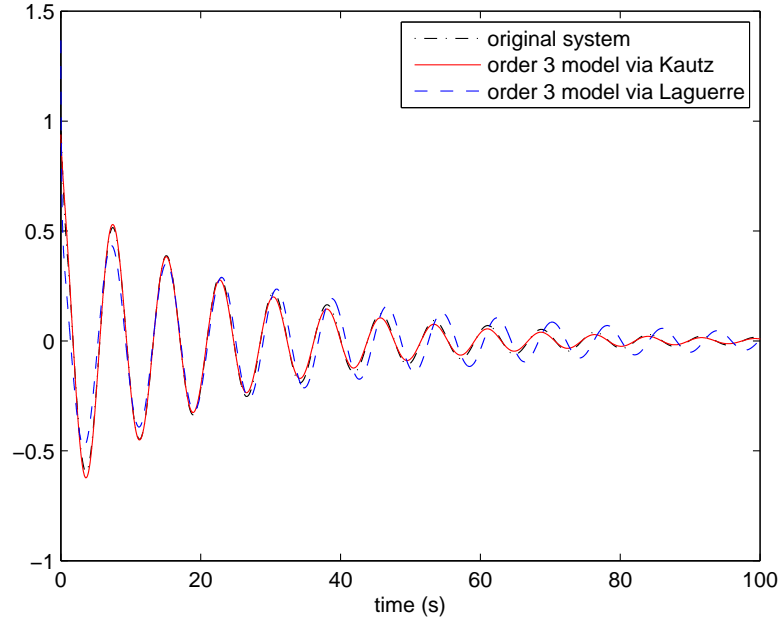


Fig.3. Impulse response of the original system and the order 3 model obtained via Kautz and Laguerre

The second example we present is the band-pass circuit in Fig.4. first proposed by Johnson [4].

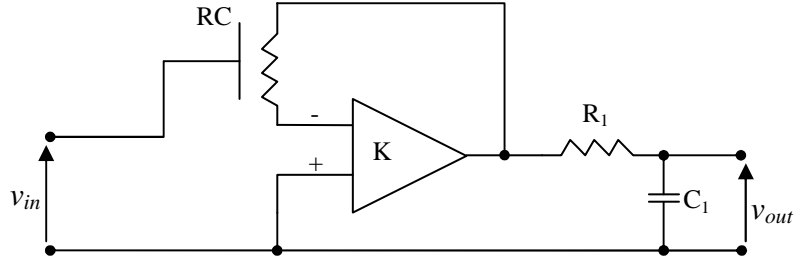


Fig.4. Band-pass circuit

This system has an irrational transfer function given by

$$\hat{g}(s) = \frac{\frac{K}{R_1 C_1} (1 - \cosh \theta)}{(\cosh \theta + K) \left(s + \frac{1}{R_1 C_1} \right)}$$

with $\theta = \sqrt{RCs}$, $RC = 18.7992$, $R_1 C_1 = 0.909087$ and $K = 9.95065$. Using 60 Kautz coefficients computed for $c = 1$ and quasi-optimal corresponding $b = 0.11454$ let us compute the 5×5 Gram matrix constructed with the inner products of $g(t)$, its first integral and its first three derivatives (i.e. $r = 4$, $q = 2$). As seen in the previous example this Gram matrix allows the derivation of a rational approximation of $\hat{g}(s)$:

$$\tilde{\hat{g}}_K(s) = \frac{-10.56s^3 - 17.96s^2 - 27.89s + 0.6894}{s^4 + 2.588s^3 + 4.351s^2 + 2.824s + 3.118}.$$

The impulse response of the original system, the reduced order 4 model given above, and the best order 4 model computed by the use of Laguerre functions instead of Kautz [11] are shown in Fig.5. The difference in accuracy is

visible, especially in Fig.6., where the variation of the squared error is plotted. Indeed the model using a Gram matrix computed via Laguerre modeling yields a relative quadratic error of $\|\varepsilon_L\|^2 = 1.35 \times 10^{-2}$, to be compared with $\|\varepsilon_K\|^2 = 1.63 \times 10^{-3}$ for computation via Kautz modeling. In fact, in order to obtain a comparable precision via Laguerre, around 200 coefficients need to be taken into consideration.

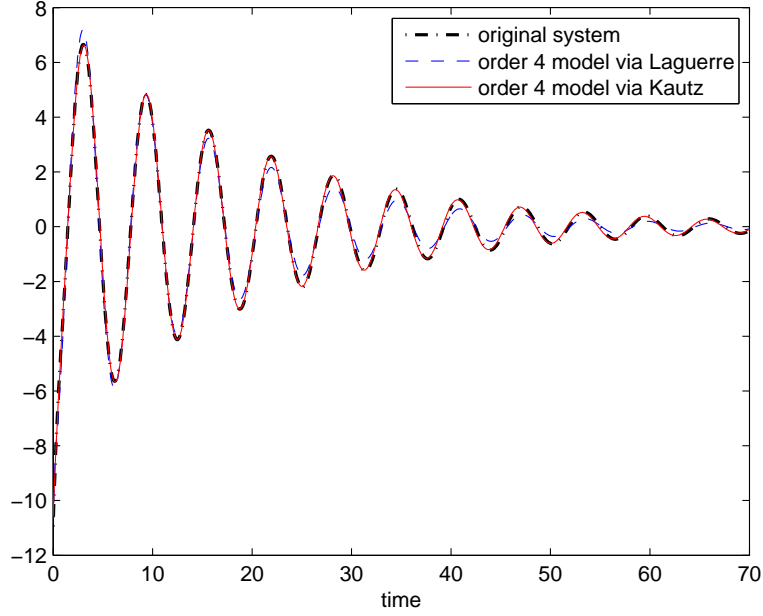


Fig.5. Impulse response of the original system, and the two reduced order models

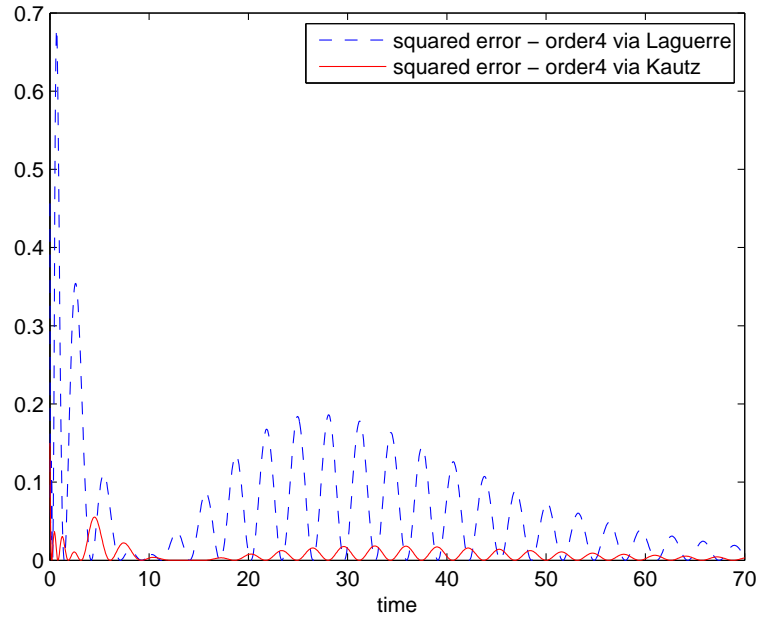


Fig.6. Squared error $\varepsilon^2(t) = [g(t) - \tilde{g}(t)]^2$

6. Conclusion

Accurately computing the Gram matrix of resonant systems may prove time consuming. The method proposed in this paper is in such cases a viable alternative, benefiting from the interesting properties of two parameter Kautz functions. The work is also relevant to the model order reduction of infinite dimensional systems

as shown by the illustrative examples.

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Appendix: Gram matrix computation algorithm

Once the Kautz spectra of the successive integrals and derivatives of the original function have been

computed using (19) or (20) and (14) or (16), the algorithm to compute the Gram matrix is similar to the one described in [11] and is summarized as follows :

1. Compute the elements of the main diagonal using (21).
2. Compute the elements of the first secondary diagonal using the property $\psi_{i,i-1} = -g_{i-1}^2(+0)/2$.

Recursively compute the other elements of the lower triangular part of the Gram matrix using the relation

$$\psi_{i,j} = -g_{i-1}(+0)g_j(+0) - \psi_{i-1,j+1}.$$

3. Using its symmetry property $\psi_{i,j} = \psi_{j,i}$ complete the matrix with its upper triangular part.

It will be noted that the initial time values $g_i(+0)$ can be evaluated using (18).